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Calculus of Variations

Non-uniqueness of weak solutions to hyperviscous Navier–Stokes equations: on sharpness of J.-L. Lions exponent

Tianwen Luo[1] · Edriss S. Titi[2,3,4]

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Abstract
Using the convex integration technique for the three-dimensional Navier–Stokes equations introduced by Buckmaster and Vicol, it is shown the existence of non-unique weak solutions for the 3D Navier–Stokes equations with fractional hyperviscosity $(-\Delta)^{\theta}$, whenever the exponent θ is less than Lions' exponent 5/4, i.e., when $\theta < 5/4$.

Mathematics Subject Classification 35Q30

1 Introduction

In this paper we consider the question of non-uniqness of weak solutions to the 3D Navier–Stokes equations with fractional viscosity (FVNSE) on $\mathbb{T}^3$

$$\begin{cases} \partial_t v + \nabla \cdot (v \otimes v) + \nabla p + \nu(-\Delta)^{\theta} v = 0, \\ \nabla \cdot v = 0, \end{cases} \tag{1}$$

where $\theta \in \mathbb{R}$ is a fixed constant, and for $u \in C^{\infty}(\mathbb{T}^3)$ with $\int_{\mathbb{T}^3} u(x)dx = 0$, the fractional Laplacian is defined via the Fourier transform as

$$\mathcal{F}((-\Delta)^{\theta} u)(\xi) = |\xi|^{2\theta} \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{Z}^3.$$

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✉ Edriss S. Titi
titi@math.tamu.edu; Edriss.Titi@damtp.cam.ac.uk; edriss.titi@weizmann.ac.il

Tianwen Luo
twluo@mail.tsinghua.edu.cn

1 Yau Mathematical Sciences Center, Tsinghua University, Beijing, China

2 Department of Mathematics, Texas A&M University, 3368 TAMU, College Station, TX 77843-3368, USA

3 Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

4 Department of Computer Science and Applied Mathematics, The Weizmann Institute of Science, 76100 Rehovot, Israel

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Definition (weak solutions) A vector field $v \in C^0_{weak}(\mathbb{R}; L^2(\mathbb{T}^3))$ is called a weak solution to the FVNSE if it solves (1) in the sense of distribution.

When $\theta = 1$, FVNSE (1) is the standard Navier–Stokes equations. Lions first considered FVNSE (1) in [20], and showed the existence and uniqueness of weak solutions to the initial value problem, which also satisfied the energy equality, for $\theta \in [5/4, \infty)$ in [21]. Moreover, an analogue of the Caffarelli–Kohn–Nirenberg [6] result was established in [18] for the FVNSE system (1), showing that the Hausdorff dimension of the singular set, in space and time, is bounded by $5 - 4\theta$ for $\theta \in (1, 5/4)$. The existence, uniqueness, regularity and stability of solutions to the FVNSE have been studied in [17, 26, 28, 29] and references therein. Very recently, using the method of convex integration introduced in [12], Colombo et al. [8] showed the non-uniquenss of Leray weak solutions to FVNSE (1) for $\theta \in (0, 1/5)$ and for $\theta \in (0, 1/3)$ in [13].

In the recent breakthrough work [5], Buckmaster and Vicol obtained non-uniqueness of weak solutions to the three-dimensional Navier–Stokes equations. They developed a new convex integration scheme in Sobolev spaces using intermittent Beltrami flows which combined concentrations and oscillations. Later, the idea of using intermittent flows was used to study non-uniqueness for transport equations in [23–25] employing scaled Mikado waves, and for stationary Navier–Stokes equations in [7, 22] employing viscous eddies.

The schemes in [5, 24] are based on the convex integration framework in Hölder spaces for the Euler equations, introduced by De Lellis and Székelyhidi [12], subsequently refined in [2, 3, 10, 15], and culminated in the proof of the second half of the Onsager conjecture by Isett in [16]; also see [4] for a shorter proof. For the first half of the Onsager conjecture, see, e.g., [1, 9], and the references therein.

The main contribution of this note is to show that the results in Buckmaster–Vicol's paper hold for FVNSE (1) for $\theta < 5/4$:

Theorem 1 Assume that $\theta \in [1, 5/4)$. Suppose u is a smooth divergence-free vector field, define on $\mathbb{R}_+ \times \mathbb{T}^3$, with compact support in time and satisfies the condition

$$\int_{\mathbb{T}^3} u(t, x)dx \equiv 0.$$

Then for any given $\varepsilon_0 > 0$, there exists a weak solution v to the FVNSE (1), with compact support in time, satisfying

$$\|v - u\|_{L^\infty_t W^{2\theta-1,1}_x} < \varepsilon_0.$$

As a consequence there are infinitely many weak solutions of the FVNSE (1) which are compactly supported in time; in particular, there are infinitely many weak solutions with initial values zero.

Remark 1 In the above theorem we assume that $\theta \in [1, 5/4)$. However, using the constructions in [5] with a slightly different choice of parameters, one can actually show that Theorem 1.2 and Theorem 1.3 in [5] hold for the 3D FVNSE, i.e., there exist non-unique weak solutions $v \in C^0_t W^{\beta,2}_x$, with a different $\beta > 0$, depending on θ. However, in this paper we choose to prove a weaker result, Theorem 1, in order to simplify the presentation while retaining the main idea.

Remark 2 For the case $\theta \in (-\infty, 1)$, the same construction also yields weak solutions $v \in C^0_t L^2_x \cap C^0_t W^{1,1}_x$ with a suitable choice of parameters.

We now make some comments on the analysis in this paper. Using the technique in [5], we adapt a convex integration scheme with intermittent Beltrami flows as the building blocks. The main difficulty in a convex integration scheme for (FVNSE), is the error induced by the frictional viscosity $\nu(-\Delta)^{\theta} v$, which is greater for a larger exponent θ. This error is controlled by making full use of the concentration effect of intermittent flows introduced in [5]. As it is shown in the crucial estimate (36), the error is controllable only for $\theta < 5/4$. Compared with [5], since our goal is to construct weak solutions $v \in C_t^0 L^2_{x,weak} \cap L_t^{\infty} W_x^{2\theta-1,1}$, we adapt a slightly simpler cut-off function and prove only estimates that are sufficient for this purpose.

2 Outline

2.1 Iteration lemma

Following [5], we consider the approximate system

$$\begin{cases} \partial_t v + \nabla \cdot (v \otimes v) + \nabla p + \nu(-\Delta)^{\theta} v = \nabla \cdot R, \\ \nabla \cdot v = 0, \end{cases} \tag{2}$$

where R is a symmetric 3×3 matrix.

Lemma 1 (Iteration Lemma for L^2 weak solutions) Let $\theta \in (-\infty, 5/4)$. Assume (v_q, R_q) is a smooth solution to (2) with

$$\|R_q\|_{L_t^{\infty} L_x^1} \leq \delta_{q+1}, \tag{3}$$

for some $\delta_{q+1} > 0$. Then for any given $\delta_{q+2} > 0$, there exists a smooth solution (v_{q+1}, R_{q+1}) of (2) with

$$\|R_{q+1}\|_{L_t^{\infty} L_x^1} \leq \delta_{q+2}, \tag{4}$$

$$\text{and} \quad \operatorname{supp}_t v_{q+1} \cup \operatorname{supp}_t R_{q+1} \subset N_{\delta_{q+1}}(\operatorname{supp}_t v_q \cup \operatorname{supp}_t R_q). \tag{5}$$

Here for a given set $A \subset \mathbb{R}$, the δ-neighborhood of A is denoted by

$$N_{\delta}(A) = \{y \in \mathbb{R} : \exists y' \in A, |y - y'| < \delta\}.$$

Furthermore, the increment $w_{q+1} = v_{q+1} - v_q$ satisfies the estimates

$$\|w_{q+1}\|_{L_t^{\infty} L_x^2} \leq C\delta_{q+1}^{1/2}, \tag{6}$$

$$\|w_{q+1}\|_{L_t^{\infty} W_x^{2\theta-1,1}} \leq \delta_{q+2}, \tag{7}$$

where the positive constant C depends only on θ.

Proof of Theorem 1 Assume Lemma 1 is valid. Let $v_0 = u$. Then

$$\int_{\mathbb{T}^3} \partial_t v_0(t,x)dx = \frac{d}{dt} \int_{\mathbb{T}^3} v_0(t,x)dx \equiv 0.$$

Let

$$R_0 = \mathcal{R}(\partial_t v_0 + \nu(-\Delta)^{\theta} v_0) + v_0 \otimes v_0 + p_0 I, \quad p_0 = -\frac{1}{3}|v_0|^2,$$

where $\mathcal{R}$ is the symmetric anti-divergence operator established in Lemma 5, below. Clearly (v_0, R_0) solves (2). Set

$$\delta_1 = \|R_0\|_{L_t^\infty L_x^1},$$
$$\delta_{q+1} = 2^{-q}\varepsilon_0, \quad \text{for } q \geq 1.$$

Apply Lemma 1 iteratively to obtain smooth solution (v_q, R_q) to (2). It follows from (6) that

$$\sum \|v_{q+1} - v_q\|_{L_t^\infty L_x^2} = \sum \|w_{q+1}\|_{L_t^\infty L_x^2} \leq C \sum \delta_{q+1}^{1/2} < \infty.$$

Thus v_q converge strongly to some $v \in C_t^0 L_x^2$. Since $\|R_{q+1}\|_{L_t^\infty L_x^1} \to 0$, as $q \to \infty$, v is a weak solution to the FVNSE (1). Estimate (7) leads to

$$\|v - v_0\|_{L_t^\infty W_x^{2\theta-1,1}} \leq \sum_{q=1}^{\infty} \|w_q\|_{L_t^\infty W_x^{2\theta-1,1}} \leq \sum_{q=1}^{\infty} \delta_{q+1} \leq \varepsilon_0.$$

Furthermore, it follows from (5) that

$$\mathrm{supp}_t v \subset \cup_{q \geq 0} \mathrm{supp}_t v_q \subset N_{\sum_{q\geq 0}\delta_{q+1}}(\mathrm{supp}_t u) \subset N_{\delta_1+\varepsilon_0}(\mathrm{supp}_t u).$$

Now we show the existence of infinitely many weak solutions with initial values zero. Let $u(t,x) = \varphi(t)\sum_{|k|\leq N} a_k e^{ik\cdot x}$ with $a_k \neq 0, a_k \cdot k = 0, a_{-k} = a_k^*$ for all $|k| \leq N$, and $\varphi \in C_c^\infty(\mathbb{R}_+)$. Thus $\nabla \cdot u = 0$ satisfies the conditions of the theorem. Hence there exists a weak solution v to (1) close enough to u so that $v \not\equiv 0$. □

3 Iteration scheme

3.1 Notations and parameters

For a complex number $\zeta \in \mathbb{C}$, we denote by ζ^* its complex conjugate. Let us normalize the volume

$$|\mathbb{T}^3| = 1.$$

For smooth functions $u \in C^\infty(\mathbb{T}^3)$ with $\int_{\mathbb{T}^3} u(x)dx = 0$ and $s \in \mathbb{R}$, we define

$$\mathcal{F}(|\nabla|^s u)(\xi) = |\xi|^s \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{Z}^3.$$

For $M, N \in [0, +\infty]$, denote the Fourier projection of u by

$$\mathcal{F}(\mathbb{P}_{[M,N)}u) = \begin{cases} u(\xi), & M \leq |\xi| < N, \xi \in \mathbb{Z}^3, \\ 0, & \text{otherwise}. \end{cases}$$

We also denote $\mathbb{P}_{\leq k} = \mathbb{P}_{[0,k)}$ and $\mathbb{P}_{\geq k} = \mathbb{P}_{[k,+\infty)}$ for $k > 0$.

Following the notation in [5], we introduce here several parameters σ, r, λ, with

$$0 < \sigma < 1 < r < \lambda < \mu < \lambda^2, \quad \sigma r < 1, \tag{8}$$

where $\lambda = \lambda_{q+1} \in 5\mathbb{N}$ is the 'frequency' parameter; σ with $1/\sigma \in \mathbb{N}$ is a small parameter such that $\lambda\sigma \in \mathbb{N}$ parameterizes the spacing between frequencies; $r \in \mathbb{N}$ denotes the number of frequencies along edges of a cube; μ measures the amount of temporal oscillation.

Later σ, r, μ will be chosen to be suitable powers of λ_{q+1}. We also fix a constant $p > 1$ which will be chosen later to be close to 1. The constants implicitly in the notation '$\lesssim$' may depend on p but are independent of the parameters σ, r, λ.

3.2 Intermittent Beltrami flows

We use intermittent Beltrami flows introduced in [5] as the building blocks. Recall some basic facts of Beltrami waves.

Proposition 1 [5, Proposition 3.1] Given $\overline{\xi} \in \mathbb{S}^2 \cap \mathbb{Q}^3$, let $A_{\overline{\xi}} \in \mathbb{S}^2 \cap \mathbb{Q}^3$ be such that

$$A_{\overline{\xi}} \cdot \overline{\xi} = 0, \quad |A_{\overline{\xi}}| = 1, \quad A_{-\overline{\xi}} = A_{\overline{\xi}}.$$

Let Λ be a given finite subset of $\mathbb{S}^2$ such that $-\Lambda = \Lambda$, and $\lambda \in \mathbb{Z}$ be such that $\lambda\Lambda \subset \mathbb{Z}^3$. Then for any choice of coefficients $a_{\overline{\xi}} \in \mathbb{C}$ with $a_{\overline{\xi}}^* = a_{-\overline{\xi}}$ the vector field

$$W(x) = \sum_{\overline{\xi} \in \Lambda} a_{\overline{\xi}} B_{\overline{\xi}} e^{i\lambda\overline{\xi}\cdot x}, \quad \text{with } B_{\overline{\xi}} = \frac{1}{\sqrt{2}}\Big(A_{\overline{\xi}} + i\overline{\xi} \times A_{\overline{\xi}}\Big),$$

is real-valued, divergence-free and satisfies

$$\nabla \times W = \lambda W, \quad \nabla \cdot (W \otimes W) = \nabla \frac{|W|^2}{2}.$$

Furthermore,

$$\langle W \otimes W \rangle := \fint_{\mathbb{T}^3} W \otimes W dx = \sum_{\overline{\xi} \in \Lambda} \frac{1}{2} |a_{(\overline{\xi})}|^2 (\mathrm{Id} - \overline{\xi} \otimes \overline{\xi}).$$

Let $\Lambda, \Lambda^+, \Lambda^- \subset \mathbb{S}^2 \cap \mathbb{Q}^3$ be defined by

$$\Lambda^+ = \left\{ \frac{1}{5}(3e_1 \pm 4e_2), \frac{1}{5}(3e_2 \pm 4e_3), \frac{1}{5}(3e_3 \pm 4e_1) \right\},$$
$$\Lambda^- = -\Lambda^+, \quad \Lambda = \Lambda^+ \cup \Lambda^-.$$

Clearly we have

$$5\Lambda \in \mathbb{Z}^3, \quad \text{and} \quad \min_{\overline{\xi}', \overline{\xi} \in \Lambda, \overline{\xi}' + \overline{\xi} \neq 0} |\overline{\xi}' + \overline{\xi}| \geq \frac{1}{5}. \tag{9}$$

Also it is direct to check that

$$\frac{1}{8} \sum_{\overline{\xi} \in \Lambda} (\mathrm{Id} - \overline{\xi} \otimes \overline{\xi}) = \mathrm{Id}.$$

In fact, representations of this form exist for symmetric matrices close to the identity. We have the following simple variant of [5, Proposition 3.2].

Proposition 2 Let $B_\varepsilon(\mathrm{Id})$ denote the ball of symmetric matrices, centered at the identity, of radius ε. Then there exist a constant $\varepsilon_\gamma > 0$ and smooth positive functions $\gamma_{(\overline{\xi})} \in C^\infty(B_{\varepsilon_\gamma}(\mathrm{Id}))$, such that

1. $\gamma_{(\overline{\xi})} = \gamma_{(-\overline{\xi})}$;
2. for each $R \in B_{\varepsilon_\gamma}(\mathrm{Id})$ we have the identity

$$R = \frac{1}{2} \sum_{\overline{\xi} \in \Lambda} \Big(\gamma_{(\overline{\xi})}(R)\Big)^2 (\mathrm{Id} - \overline{\xi} \otimes \overline{\xi}).$$

Define the Dirichlet kernel

$$D_r(x) = \frac{1}{(2r+1)^{3/2}} \sum_{\xi \in \Omega_r} e^{i\xi \cdot x}, \quad \Omega_r = \{(j,k,l) : j,k,l \in \{-r,\dots,r\}\}.$$

It has the property that, for $1 < p \le \infty$,

$$\|D_r\|_{L^p} \lesssim r^{3/2-3/p}, \quad \|D_r\|_{L^2} = (2\pi)^3.$$

Following [5], for $\overline{\xi} \in \Lambda^+$, define a directed and rescaled Dirichlet kernel by

$$\eta_{(\overline{\xi})}(t,x) = \eta_{\overline{\xi},\lambda,\sigma,r,\mu}(t,x) = D_r(\lambda\sigma(\overline{\xi} \cdot x + \mu t, A_{\overline{\xi}} \cdot x, (\overline{\xi} \times A_{\overline{\xi}}) \cdot x)), \tag{10}$$

and for $\overline{\xi} \in \Lambda^-$, define

$$\eta_{(\overline{\xi})}(t,x) = \eta_{-(\overline{\xi})}(t,x).$$

Note the important identity

$$\frac{1}{\mu}\partial_t \eta_{(\overline{\xi})}(t,x) = \pm(\overline{\xi} \cdot \nabla)\eta_{(\overline{\xi})}(t,x), \quad \overline{\xi} \in \Lambda^{\pm}. \tag{11}$$

Since the map $x \mapsto \lambda\sigma(\overline{\xi} \cdot x + \mu t, A_{\overline{\xi}} \cdot x, (\overline{\xi} \times A_{\overline{\xi}}) \cdot x)$ is the composition of a rotation by a rational orthogonal matrix mapping $\{e_1, e_2, e_3\}$ to $\{\overline{\xi}, A_{\overline{\xi}}, \overline{\xi} \times A_{\overline{\xi}}\}$, a translation, and a rescaling by integers, for $1 < p \le \infty$, we have

$$\fint_{\mathbb{T}^3} \eta_{(\overline{\xi})}(t,x)^2(t,x)dx = 1, \quad \|\eta_{(\overline{\xi})}\|_{L^\infty_t L^p_x(\mathbb{T}^3)} \lesssim r^{3/2-3/p}.$$

Let $W_{(\overline{\xi})}$ be the Beltrami plane wave at frequency λ,

$$W_{(\overline{\xi})} = W_{\overline{\xi},\lambda}(x) = B_{\overline{\xi}} e^{i\lambda\overline{\xi}\cdot x}.$$

Define the intermittent Beltrami wave $\mathbb{W}_{(\overline{\xi})}$ as

$$\mathbb{W}_{(\overline{\xi})}(t,x) := \mathbb{W}_{\overline{\xi},\lambda,\sigma,r,\mu}(t,x) = \eta_{(\overline{\xi})}(t,x) W_{(\overline{\xi})}(x). \tag{12}$$

It follows from the definitions and (9) that

$$\mathbb{P}_{[\frac{\lambda}{2},2\lambda)} \mathbb{W}_{(\overline{\xi})} = \mathbb{W}_{(\overline{\xi})}, \tag{13}$$

$$\mathbb{P}_{[\frac{\lambda}{5},4\lambda)} \Big(\mathbb{W}_{(\overline{\xi})} \otimes \mathbb{W}_{(\overline{\xi}')}\Big) = \mathbb{W}_{(\overline{\xi})} \otimes \mathbb{W}_{(\overline{\xi}')}, \quad \overline{\xi}' \ne -\overline{\xi}. \tag{14}$$

The following properties are immediate from the definitions.

Proposition 3 [5, Proposition 3.4] Let $a_{\overline{\xi}} \in \mathbb{C}$ be constants with $a^*_{\overline{\xi}} = a_{-\overline{\xi}}$. Let

$$W(x) = \sum_{\overline{\xi} \in \Lambda} a_{\overline{\xi}} \mathbb{W}_{(\overline{\xi})}(x).$$

Then $W(x)$ is real valued. Moreover, for each $R \in B_{\varepsilon_\gamma}(\mathrm{Id})$ we have

$$\sum_{\overline{\xi} \in \Lambda} \Big(\gamma_{(\overline{\xi})}(R)\Big)^2 \fint_{\mathbb{T}^3} \mathbb{W}_{(\overline{\xi})} \otimes \mathbb{W}_{(-\overline{\xi})} = \sum_{\overline{\xi} \in \Lambda} \Big(\gamma_{(\overline{\xi})}(R)\Big)^2 B_{\overline{\xi}} \otimes B_{-\overline{\xi}} = R.$$

Proposition 4 [5, Proposition 3.5] For any $1 < p \le \infty$, $N \ge 0$, $K \ge 0$:

$$\Big\|\nabla^N \partial_t^K \mathbb{W}_{(\overline{\xi})}\Big\|_{L^\infty_t L^p_x} \lesssim \lambda^N (\lambda\sigma r\mu)^K r^{3/2-3/p}, \tag{15}$$

$$\Big\|\nabla^N \partial_t^K \eta_{(\overline{\xi})}\Big\|_{L^\infty_t L^p_x} \lesssim (\lambda\sigma r)^N (\lambda\sigma r\mu)^K r^{3/2-3/p}. \tag{16}$$

3.3 Perturbations

Let $\psi(t)$ be a smooth cut-off function such that

$$\psi(t) = 1 \text{ on } \operatorname{supp}_t R_q, \quad \operatorname{supp} \psi(t) \subset N_{\delta_{q+1}}(\operatorname{supp}_t R_q), \quad |\psi'(t)| \leq 2\delta_{q+1}^{-1}. \tag{17}$$

Take a smooth increasing function χ such that

$$\chi(s) = \begin{cases} 1, & 0 \leq s < 1 \\ s, & s \geq 2 \end{cases},$$

and set

$$\rho(t,x) = \varepsilon_\gamma^{-1} \delta_{q+1} \chi\left(\delta_{q+1}^{-1} |R_q(t,x)|\right) \psi^2(t).$$

where ε_γ is the constant in Proposition 2. Then clearly

$$\operatorname{supp}_t \rho \subset N_{\delta_{q+1}}(\operatorname{supp}_t R_q). \tag{18}$$

It follows from the above definition that

$$|R_q|/\rho = \varepsilon_\gamma \frac{|R_q|}{\delta_{q+1}\chi\left(\delta_{q+1}^{-1}|R_q(t,x)|\right)\psi^2} \leq \varepsilon_\gamma \implies \mathrm{Id} - R_q/\rho \in B_{\varepsilon_\gamma}(\mathrm{Id}) \text{ on } \operatorname{supp} R_q.$$

Therefore, the amplitude functions

$$a_{(\bar{\xi})}(t,x) := \rho^{1/2}(t,x)\gamma_{(\bar{\xi})}(\mathrm{Id} - \rho(t,x)^{-1} R_q(t,x))$$

are well-defined and smooth. Define the velocity perturbation to be $w = w_{q+1}$:

$$\begin{aligned}
w &= w^{(p)} + w^{(c)} + w^{(t)}, \\
w^{(p)} &= \sum_{\bar{\xi}\in\Lambda} a_{(\bar{\xi})} \mathbb{W}_{(\bar{\xi})} = \sum_{\bar{\xi}\in\Lambda} a_{(\bar{\xi})}(t,x)\eta_{(\bar{\xi})}(t,x) B_{\bar{\xi}} e^{i\lambda\bar{\xi}\cdot x}, \\
w^{(c)} &= \frac{1}{\lambda_{q+1}} \sum_{\bar{\xi}\in\Lambda} \nabla\left(a_{(\bar{\xi})}\eta_{(\bar{\xi})}\right) \times W_{(\bar{\xi})}, \\
w^{(t)} &= \frac{1}{\mu} \sum_{\bar{\xi}\in\Lambda^+} \mathbb{P}_{LH}\mathbb{P}_{\neq 0}\left(a_{(\bar{\xi})}^2 \eta_{(\bar{\xi})}^2 \bar{\xi}\right),
\end{aligned}$$

where $\mathbb{P}_{LH} = \mathrm{Id} - \nabla\Delta^{-1}\mathrm{div}$ is the Leray-Helmholtz projection into divergence-free vector field, and $\mathbb{P}_{\neq 0} f = f - \fint_{\mathbb{T}^3} f dx$. It is well-known that $\mathbb{P}_{LH}$ is bounded on L^p, $1 < p < \infty$ (see, e.g., [14]). It follows from Proposition 3 that

$$\sum_{\bar{\xi}\in\Lambda} a_{(\bar{\xi})}^2 \fint_{\mathbb{T}^3} \mathbb{W}_{(\bar{\xi})} \otimes \mathbb{W}_{(-\bar{\xi})} dx = \rho\mathrm{Id} - R_q. \tag{19}$$

3.4 Estimates for perturbations

Lemma 2 The following bounds hold:

$$\|\rho\|_{L_t^\infty L_x^1} \leq C\delta_{q+1}, \tag{20}$$

$$\|\rho^{-1}\|_{C^0(\operatorname{supp} R_q)} \lesssim \delta_{q+1}^{-1}, \tag{21}$$

$$\|\rho\|_{C^N_{t,x}} \le C(\delta_{q+1}, \|R_q\|_{C^N}), \tag{22}$$

$$\|a_{(\bar\xi)}\|_{L^\infty_t L^2_x} \lesssim \|\rho\|^{1/2}_{L^\infty_t L^1_x} \lesssim \delta^{1/2}_{q+1}, \tag{23}$$

$$\|a_{(\bar\xi)}\|_{C^N_{t,x}} \le C(\delta_{q+1}, \|R_q\|_{C^N}). \tag{24}$$

Proof It follows from (3) that

$$\begin{aligned}\|\rho(t,\cdot)\|_{L^1_x} &= \int_{|R_q|\le\delta_{q+1}} \rho + \int_{|R_q|>\delta_{q+1}} \rho \lesssim \delta_{q+1} + \int_{|R_q|>\delta_{q+1}} |R_q| \\ &\le C\delta_{q+1}.\end{aligned}$$

It is direct to verify (21) and (23), while (22) and (24) follow from (17) and (21). □

Now we can estimate the time support of w_{q+1}:

$$\mathrm{supp}_t\, w_{q+1} \subset \mathrm{supp}_t\,\rho \subset \mathrm{supp}\,\psi \subset N_{\delta_{q+1}}(\mathrm{supp}_t\, R_q). \tag{25}$$

We need the following Lemma, which is a variant of [5, Lemma 3.6].

Lemma 3 ([24, Lemma 2.1]) Let $f, g \in C^\infty(\mathbb{T}^3)$, and g is $(\mathbb{T}/N)^3$ periodic, $N \in \mathbb{N}$. Then for $1 \le p \le \infty$,

$$\|fg\|_{L^p} \le \|f\|_{L^p}\|g\|_{L^p} + C_p N^{-1/p}\|f\|_{C^1}\|g\|_{L^p}.$$

Let us denote

$$\mathcal{C}_N = C\left(\sup_{\bar\xi\in\Lambda}\|a_{(\bar\xi)}\|_{C^N_{t,x}}\right) \tag{26}$$

to be some polynomials depending on $\sup_{\bar\xi\in\Lambda}\|a_{(\bar\xi)}\|_{C^N_{t,x}}$.

Lemma 4 Suppose the parameters satisfy (8) and

$$r^{3/2} \le \mu. \tag{27}$$

Then the following estimates for the perturbations hold:

$$\left\|w^{(p)}_{q+1}\right\|_{L^\infty_t L^2_x} \lesssim \delta^{1/2}_{q+1} + (\lambda_{q+1}\sigma)^{-1/2}\mathcal{C}_1, \tag{28}$$

$$\|w_{q+1}\|_{L^\infty_t L^p_x} \lesssim r^{3/2-3/p}\mathcal{C}_1, \tag{29}$$

$$\left\|w^{(c)}_{q+1}\right\|_{L^\infty_t L^p_x} + \left\|w^{(t)}_{q+1}\right\|_{L^\infty_t L^p_x} \lesssim (\sigma r + \mu^{-1}r^{3/2})r^{3/2-3/p}\mathcal{C}_1, \tag{30}$$

$$\left\|\partial_t w^{(p)}_{q+1}\right\|_{L^\infty_t L^p_x} + \left\|\partial_t w^{(c)}_{q+1}\right\|_{L^\infty_t L^p_x} \lesssim \lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_2, \tag{31}$$

$$\||\nabla|^N w_{q+1}\|_{L^\infty_t L^p_x} \lesssim r^{3/2-3/p}\lambda^N_{q+1}\mathcal{C}_{N+1}, \tag{32}$$

for $1 < p < \infty$, $N \ge 1$.

Proof Since $\mathbb{W}_{(\bar\xi)}$ is $(\mathbb{T}/\lambda\sigma)^3$ periodic, it follows from (15), (23), and Lemma 3 that

$$\begin{aligned}\left\|w^{(p)}_{q+1}\right\|_{L^\infty_t L^2_x} &\lesssim \sum_{\bar\xi\in\Lambda}\left(\left\|a_{(\bar\xi)}\right\|_{L^\infty_t L^2_x} + (\lambda_{q+1}\sigma)^{-1/2}\left\|a_{(\bar\xi)}\right\|_{C^1}\right)\|\mathbb{W}_{(\bar\xi)}\|_{L^\infty_t L^2_x} \\ &\lesssim \delta^{1/2}_{q+1} + (\lambda_{q+1}\sigma)^{-1/2}\mathcal{C}_1.\end{aligned}$$

In view of (8), (15) and (16) yield that

$$
\begin{aligned}
\left\| w_{q+1}^{(p)} \right\|_{L_t^\infty L_x^p} &\lesssim \sum_{\overline{\xi}\in\Lambda} \left\| a_{(\overline{\xi})} \right\|_{C^0} \left\| \mathbb{W}_{(\overline{\xi})} \right\|_{L_t^\infty L_x^p} \lesssim r^{3/2-3/p}\mathcal{C}_0, \\
\left\| w_{q+1}^{(c)} \right\|_{L_t^\infty L_x^p} &\lesssim \lambda_{q+1}^{-1} \sum_{\overline{\xi}\in\Lambda} \left(\left\| \eta_{(\overline{\xi})} \right\|_{L_t^\infty L_x^p} + \left\| \nabla \eta_{(\overline{\xi})} \right\|_{L_t^\infty L_x^p} \right) \left\| a_{(\overline{\xi})} \right\|_{C^1} \left\| \mathbb{W}_{(\overline{\xi})} \right\|_{L_t^\infty L_x^p} \\
&\lesssim (\sigma r) r^{3/2-3/p}\mathcal{C}_1, \\
\left\| w_{q+1}^{(t)} \right\|_{L_t^\infty L_x^p} &\lesssim \mu^{-1} \sum_{\overline{\xi}\in\Lambda^+} \left\| a_{(\overline{\xi})}^2 \eta_{(\overline{\xi})}^2 \overline{\xi} \right\|_{L_t^\infty L_x^p} \lesssim \mu^{-1} \sum_{\overline{\xi}\in\Lambda^+} \left\| a_{(\overline{\xi})}^2 \right\|_{C^0} \left\| \eta_{(\overline{\xi})} \right\|_{L_t^\infty L_x^{2p}}^2 \\
&\lesssim \mu^{-1} r^{3-3/p}\mathcal{C}_0,
\end{aligned}
$$

where the boundedness of $\mathbb{P}_{LH}$ and $\mathbb{P}_{\neq 0}$ on L^p, for $1 < p < \infty$, is used in the first inequality of the estimate for $\|w_{q+1}^{(t)}\|_{L_t^\infty L_x^p}$. In the same way, we can estimate

$$
\begin{aligned}
\left\| \partial_t w_{q+1}^{(p)} \right\|_{L_t^\infty L_x^p} &\lesssim \sum_{\overline{\xi}\in\Lambda} \left\| \partial_t a_{(\overline{\xi})} \right\|_{C^0} \left\| \mathbb{W}_{(\overline{\xi})} \right\|_{L_t^\infty L_x^p} + \left\| a_{(\overline{\xi})} \right\|_{C^0} \left\| \partial_t \mathbb{W}_{(\overline{\xi})} \right\|_{L_t^\infty L_x^p} \\
&\lesssim \lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_1, \\
\left\| \partial_t w_{q+1}^{(c)} \right\|_{L_t^\infty L_x^p} &\lesssim \lambda_{q+1}^{-1} \sum_{\overline{\xi}\in\Lambda} \left\| a_{(\overline{\xi})} \right\|_{C_{t,x}^2} \Bigg(\left\| \eta_{(\overline{\xi})} \right\|_{L_t^\infty L_x^p} + \left\| \nabla \eta_{(\overline{\xi})} \right\|_{L_t^\infty L_x^p} + \left\| \partial_t \eta_{(\overline{\xi})} \right\|_{L_t^\infty L_x^p} \\
&\quad + \left\| \partial_t \nabla \eta_{(\overline{\xi})} \right\|_{L_t^\infty L_x^p} \Bigg) \lesssim \sigma r \lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_2 \lesssim \lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_2.
\end{aligned}
$$

For $N \geq 1$, using (15) and (16), we obtain that

$$
\begin{aligned}
\left\| \nabla^N w_{q+1}^{(p)} \right\|_{L_t^\infty L_x^p} &\lesssim \sum_{\overline{\xi}\in\Lambda} \sum_{k=0}^{N} \left\| \nabla^k a_{(\overline{\xi})} \right\|_{C^0} \left\| \nabla^{N-k} \mathbb{W}_{(\overline{\xi})} \right\|_{L_t^\infty L_x^p} \\
&\lesssim \lambda_{q+1}^N r^{3/2-3/p}\mathcal{C}_N, \\
\left\| \nabla^N w_{q+1}^{(c)} \right\|_{L_t^\infty L_x^p} &\lesssim \lambda_{q+1}^{-1} \sum_{\overline{\xi}\in\Lambda} \sum_{m=0}^{N} \sum_{k=0}^{m} \lambda_{q+1}^{N-m} \left\| \nabla^{k+1} a_{(\overline{\xi})} \right\|_{C^0} \left\| \nabla^{m-k} \eta_{(\overline{\xi})} \right\|_{L_t^\infty L_x^p} \\
&\quad + \lambda_{q+1}^{-1} \sum_{\overline{\xi}\in\Lambda} \sum_{m=0}^{N} \sum_{k=0}^{m} \lambda_{q+1}^{N-m} \left\| \nabla^{k} a_{(\overline{\xi})} \right\|_{C^0} \left\| \nabla^{m-k+1} \eta_{(\overline{\xi})} \right\|_{L_t^\infty L_x^p} \\
&\lesssim \lambda_{q+1}^N r^{3/2-3/p}\mathcal{C}_{N+1}, \\
\left\| \nabla^N w_{q+1}^{(t)} \right\|_{L_t^\infty L_x^p} &\lesssim \mu^{-1} \sum_{\overline{\xi}\in\Lambda} \sum_{m=0}^{N} \left\| \nabla^{N-m} \left(a_{(\overline{\xi})}^2 \right) \right\|_{C^0} \sum_{k=0}^{m} \left\| \nabla^k \eta_{(\overline{\xi})} \right\|_{L_t^\infty L_x^{2p}} \left\| \nabla^{m-k} \eta_{(\overline{\xi})} \right\|_{L_t^\infty L_x^{2p}} \\
&\lesssim \lambda_{q+1}^N r^{3/2-3/p} \frac{(\sigma r)^N r^{3/2}}{\mu} \mathcal{C}_N \lesssim \lambda_{q+1}^N r^{3/2-3/p}\mathcal{C}_N,
\end{aligned}
$$

where we use (8) and (27). □

3.5 Estimates for the stress

Let us recall the following operator in [12].

Lemma 5 (symmetric anti-divergence) There exists a linear operator $\mathcal{R}$, of order -1, mapping vector fields to symmetric matrices such that

$$\nabla \cdot \mathcal{R}(u) = u - \fint_{\mathbb{T}^3} u, \tag{33}$$

with standard Calderon–Zygmund estimates, for $1 < p < \infty$,

$$\|\mathcal{R}\|_{L^p \to W^{1,p}} \lesssim 1, \quad \|\mathcal{R}\|_{C^0 \to C^0} \lesssim 1, \quad \|\mathcal{R}\mathbb{P}_{\neq 0} u\|_{L^p} \lesssim \||\nabla|^{-1}\mathbb{P}_{\neq 0} u\|_{L^p}. \tag{34}$$

Proof Suppose $u \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ is a smooth vector field. Define

$$\mathcal{R}(u) = \frac{1}{4}\Big(\nabla \mathbb{P}_{LH} v + (\nabla \mathbb{P}_{LH} v)^T\Big) + \frac{3}{4}\Big(\nabla v + (\nabla v)^T\Big) - \frac{1}{2}(\nabla \cdot v)\mathrm{Id}$$

where $v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ is the unique solution to $\Delta v = u - \fint_{\mathbb{T}^3} u$ with $\fint_{\mathbb{T}^3} v = 0$.

It is direct to verify that $\mathcal{R}(u)$ is a symmetric matrix field depending linearly on u and satisfies (33). Note that $\mathcal{R}$ is a constant coefficient ellitpic operator of order -1. We refer to [14] for the Calderon-Zygmund estimates $\|\mathcal{R}\|_{L^p \to W^{1,p}} \lesssim 1$ and $\|\mathcal{R}\mathbb{P}_{\neq 0} u\|_{L^p} \lesssim \||\nabla|^{-1}\mathbb{P}_{\neq 0} u\|_{L^p}$. Combining these with Sobolev embeddings, we have $\|\mathcal{R}u\|_{C^\alpha} \lesssim \|\mathcal{R}u\|_{W^{1,4}} \lesssim \|u\|_{L^4} \lesssim \|u\|_{C^0}$, with $\alpha = 1/4$. □

We have the following variant of [5, Lemma B.1] in [5].

Lemma 6 Let $a \in C^2(\mathbb{T}^3)$. For $1 < p < \infty$, and any smooth function $f \in L^p(\mathbb{T}^3)$, we have

$$\||\nabla|^{-1}\mathbb{P}_{\neq 0}(a\mathbb{P}_{\geq k} f)\|_{L^p(\mathbb{T}^3)} \lesssim k^{-1}\|\nabla^2 a\|_{L^\infty(\mathbb{T}^3)}\|f\|_{L^p(\mathbb{T}^3)}. \tag{35}$$

Proof of Lemma 6 We follow the proof in [5]. Note that

$$|\nabla|^{-1}\mathbb{P}_{\neq 0}(a\mathbb{P}_{\geq k} f) = |\nabla|^{-1}\mathbb{P}_{\geq k/2}(\mathbb{P}_{\leq k/2} a\mathbb{P}_{\geq k} f) + |\nabla|^{-1}\mathbb{P}_{\neq 0}(\mathbb{P}_{\geq k/2} a\mathbb{P}_{\geq k} f).$$

As direct consequences of the Littlewood–Paley decomposition and Schauder estimates we have the bounds for $1 < p < \infty$ (see, for example, [14])

$$\|\mathbb{P}_{\leq k/2}\|_{L^p \to L^p} \lesssim 1, \quad \||\nabla|^{-1}\mathbb{P}_{\geq k/2}\|_{L^p \to L^p} \lesssim k^{-1}, \quad \||\nabla|^{-1}\mathbb{P}_{\neq 0}\|_{L^p \to L^p} \lesssim 1.$$

Combining these bounds with Hölder's inequality and the embedding $W^{1,4}(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$, we obtain

$$\begin{aligned}
\||\nabla|^{-1}\mathbb{P}_{\neq 0}(a\mathbb{P}_{\geq k} f)\|_{L^p} &\lesssim k^{-1}\|\mathbb{P}_{\leq k/2} a\mathbb{P}_{\geq k} f\|_{L^p} + \|\mathbb{P}_{\geq k/2} a\mathbb{P}_{\geq k} f\|_{L^p} \\
&\lesssim k^{-1}(\|\mathbb{P}_{\leq k/2} a\|_{L^\infty} + k\|\mathbb{P}_{\geq k/2} a\|_{L^\infty})\|f\|_{L^p} \\
&\lesssim k^{-1}(\|\nabla \mathbb{P}_{\leq k/2} a\|_{L^4} + k\|\nabla \mathbb{P}_{\geq k/2} a\|_{L^4})\|f\|_{L^p} \\
&\lesssim k^{-1}(\|\mathbb{P}_{\leq k/2} \nabla a\|_{L^4} + k\||\nabla|^{-1}\mathbb{P}_{\geq k/2}|\nabla|\nabla \mathbb{P}_{\geq k/2} a\|_{L^4})\|f\|_{L^p} \\
&\lesssim k^{-1}(\|\nabla a\|_{L^4} + \|\nabla^2 \mathbb{P}_{\geq k/2} a\|_{L^4})\|f\|_{L^p} \lesssim k^{-1}\|\nabla^2 a\|_{L^4}\|f\|_{L^p}.
\end{aligned}$$

□

It follows from the definition of w_{q+1} that

$$\int_{\mathbb{T}^3} w_{q+1} dx = \int_{\mathbb{T}^3} \frac{1}{\lambda_{q+1}} \sum_{\overline{\xi}\in\Lambda} \nabla\Big(a_{(\overline{\xi})}\eta_{(\overline{\xi})}W_{(\overline{\xi})}\Big) dx$$
$$+ \int_{\mathbb{T}^3} \frac{1}{\mu} \sum_{\overline{\xi}\in\Lambda^+} P_{LH}\mathbb{P}_{\neq 0}\Big(a^2_{(\overline{\xi})}\eta^2_{(\overline{\xi})}\overline{\xi}\Big) dx = 0.$$

Hence $\int_{\mathbb{T}^3} \nu(-\Delta)^\theta w_{q+1} dx = 0$ and $\frac{d}{dt}\int_{\mathbb{T}^3} w_{q+1} dx = 0$. We obtain R_{q+1} by plugging $v_{q+1} = v_q + w_{q+1}$ in (2), using (33) and the assumption that (v_q, R_q) solves (2):

$$\begin{aligned}
\nabla\cdot R_{q+1} &= \nabla\cdot\Big[\mathcal{R}\Big(\nu(-\Delta)^\theta w_{q+1} + \partial_t w^{(p)}_{q+1} + \partial_t w^{(c)}_{q+1}\Big) + v_q\otimes w_{q+1} + w_{q+1}\otimes v_q\Big] \\
&\quad + \nabla\cdot\Big[\Big(w^{(c)}_{q+1} + w^{(t)}_{q+1}\Big)\otimes w_{q+1} + w^{(p)}_{q+1}\otimes\Big(w^{(c)}_{q+1} + w^{(t)}_{q+1}\Big)\Big] \\
&\quad \times\Big[\nabla\cdot\Big(w^{(p)}_{q+1}\otimes w^{(p)}_{q+1} - R_q\Big) + \partial_t w^{(t)}_{q+1}\Big] + \nabla(p_{q+1} - p_q) \\
&:= \nabla\cdot(\widetilde{R}_{linear} + \widetilde{R}_{corrector} + \widetilde{R}_{oscillation}) + \nabla(p_{q+1} - p_q).
\end{aligned}$$

It follows from Lemma 4 that

$$\begin{aligned}
\|\widetilde{R}_{corrector}\|_{L^\infty_t L^p_x} &\lesssim \Big(\Big\|w^{(c)}_{q+1}\Big\|_{L^\infty_t L^{2p}_x} + \Big\|w^{(t)}_{q+1}\Big\|_{L^\infty_t L^{2p}_x}\Big)\Big(\|w_{q+1}\|_{L^\infty_t L^{2p}_x} + \Big\|w^{(p)}_{q+1}\Big\|_{L^\infty_t L^{2p}_x}\Big) \\
&\lesssim (\sigma r + \mu^{-1} r^{3/2}) r^{3-3/p}\mathcal{C}_1.
\end{aligned}$$

Noting that $\nabla\times\dfrac{w^{(p)}_{q+1}}{\lambda_{q+1}} = w^{(p)}_{q+1} + w^{(c)}_{q+1}$, Lemma 4 and (34) yield that

$$\begin{aligned}
&\|\widetilde{R}_{linear}\|_{L^\infty_t L^p_x} \\
&\lesssim \lambda^{-1}_{q+1}\Big\|\partial_t\mathcal{R}\nabla\times(w^{(p)}_{q+1})\Big\|_{L^\infty_t L^p_x} + \Big\|\mathcal{R}\Big(\nu(-\Delta)^\theta w_{q+1}\Big)\Big\|_{L^\infty_t L^p_x} \\
&\quad + \|v_q\otimes w_{q+1} + w_{q+1}\otimes v_q\|_{L^\infty_t L^p_x} \\
&\lesssim \lambda^{-1}_{q+1}\Big\|\partial_t w^{(p)}_{q+1}\Big\|_{L^\infty_t L^p_x} + \Big\||\nabla|^{2\theta-1} w_{q+1}\Big\|_{L^\infty_t L^p_x} + \|v_q\|_{C^0}\|w_{q+1}\|_{L^\infty_t L^p_x} \\
&\lesssim \sigma\mu r^{5/2-3/p}\mathcal{C}_2 + r^{3/2-3/p}\Big(\lambda^{2\theta-1}_{q+1} + \|v_q\|_{C^0}\Big)\mathcal{C}_3.
\end{aligned} \tag{36}$$

This is the crucial estimate to control the fractional viscosity. If we assume that $p\sim 1, r\sim\lambda^{-1}_{q+1}$, we must have $\theta < 5/4$ in order that the second term in (36) is small for λ_{q+1} sufficiently large.

It remains to estimate $\widetilde{R}_{oscillation}$, which can be handled in the same way as in [5]. It follows from (19) that

$$\begin{aligned}
\nabla\cdot\Big(w^{(p)}_{q+1}\otimes w^{(p)}_{q+1} - R_q\Big) &= \nabla\cdot\Bigg(\sum_{\overline{\xi},\overline{\xi}'\in\Lambda} a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{W}_{\overline{\xi}}\otimes\mathbb{W}_{(\overline{\xi}')} - R_q\Bigg) \\
&= \nabla\cdot\Bigg(\sum_{\overline{\xi},\overline{\xi}'\in\Lambda} a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}\Bigg) + \nabla\rho \\
&:= \sum_{\overline{\xi},\overline{\xi}'\in\Lambda} E_{(\overline{\xi},\overline{\xi}')} + \nabla\rho.
\end{aligned}$$

Since $E_{(\overline{\xi},\overline{\xi}')}$ has zero mean, we can split it as

$$\begin{aligned}E_{(\overline{\xi},\overline{\xi}')}+E_{(\overline{\xi}',\overline{\xi})}&=\mathbb{P}_{\neq 0}\Big(\nabla\Big(a_{(\overline{\xi})}a_{(\overline{\xi}')}\Big)\cdot\Big(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}\Big(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}+\mathbb{W}_{\overline{\xi}'}\otimes\mathbb{W}_{(\overline{\xi})}\Big)\Big)\Big)\\&\quad+\mathbb{P}_{\neq 0}\Big(a_{(\overline{\xi})}a_{(\overline{\xi}')}\nabla\cdot\Big(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}+\mathbb{W}_{\overline{\xi}'}\otimes\mathbb{W}_{(\overline{\xi})}\Big)\Big)\\&:=E_{(\overline{\xi},\overline{\xi}',1)}+E_{(\overline{\xi},\overline{\xi}',2)}.\end{aligned}$$

Using (15), (34) and (35), we obtain

$$\begin{aligned}\Big\|\mathcal{R}E_{(\overline{\xi},\overline{\xi}',1)}\Big\|_{L^\infty_tL^p_x}&\lesssim\Big\||\nabla|^{-1}E_{(\overline{\xi},\overline{\xi}',1)}\Big\|_{L^\infty_tL^p_x}\\&\lesssim(\lambda_{q+1}\sigma)^{-1}\Big\|a_{(\overline{\xi})}a_{(\overline{\xi}')}\Big\|_{C^3}\Big\|\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}\Big\|_{L^\infty_tL^p_x}\\&\lesssim(\lambda_{q+1}\sigma)^{-1}\Big\|a_{(\overline{\xi})}a_{(\overline{\xi}')}\Big\|_{C^3}\Big\|\mathbb{W}_{(\overline{\xi})}\Big\|_{L^\infty_tL^{2p}_x}\Big\|\mathbb{W}_{(\overline{\xi}')}\Big\|_{L^\infty_tL^{2p}_x}\\&\lesssim(\lambda_{q+1}\sigma)^{-1}r^{3-3/p}\mathcal{C}_3.\end{aligned}$$

Recall the vector identity $A\cdot\nabla B+B\cdot\nabla A=\nabla(A\cdot B)-A\times(\nabla\times B)-B\times(\nabla\times A)$. For $\overline{\xi},\overline{\xi}'\in\Lambda$, using the anti-symmetry of the cross product, we can write

$$\begin{aligned}&\nabla\cdot\Big(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}+\mathbb{W}_{(\overline{\xi}')}\otimes\mathbb{W}_{(\overline{\xi})}\Big)\\&=\Big(W_{(\overline{\xi})}\otimes W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\otimes W_{(\overline{\xi})}\Big)\nabla\Big(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\Big)\\&\quad+\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\Big(W_{(\overline{\xi})}\cdot\nabla W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\cdot\nabla W_{(\overline{\xi})}\Big)\\&=\Big(W_{(\overline{\xi}')}\cdot\nabla\Big(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\Big)\Big)W_{(\overline{\xi})}+\Big(W_{(\overline{\xi})}\cdot\nabla\Big(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\Big)\Big)W_{\overline{\xi}'}\\&\quad+\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\nabla\Big(W_{(\overline{\xi})}\cdot W_{(\overline{\xi}')}\Big).\end{aligned}$$

For the term $E_{(\overline{\xi},\overline{\xi}',2)}$, first consider the case $\overline{\xi}+\overline{\xi}'\neq 0$. It follows from the above identity and (14) that

$$\begin{aligned}&a_{(\overline{\xi})}a_{(\overline{\xi}')}\nabla\cdot\Big(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}+\mathbb{W}_{(\overline{\xi}')}\otimes\mathbb{W}_{(\overline{\xi})}\Big)\\&=a_{(\overline{\xi})}a_{(\overline{\xi}')}\nabla\cdot\mathbb{P}_{\geq\lambda_{q+1}/10}\Big(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\Big(W_{(\overline{\xi})}\otimes W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\otimes W_{(\overline{\xi})}\Big)\Big)\\&=a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\Big(\nabla\Big(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\Big)\cdot\Big(W_{(\overline{\xi})}\otimes W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\otimes W_{(\overline{\xi})}\Big)\Big)\\&\quad+a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\Big(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\nabla\Big(W_{(\overline{\xi})}\cdot W_{(\overline{\xi}')}\Big)\Big)\\&=a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\Big(\nabla\Big(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\Big)\cdot\Big(W_{(\overline{\xi})}\otimes W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\otimes W_{(\overline{\xi})}\Big)\Big)\\&\quad+\nabla\Big(a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{W}_{(\overline{\xi})}\cdot\mathbb{W}_{(\overline{\xi}')}\Big)-\nabla\Big(a_{(\overline{\xi})}a_{(\overline{\xi}')}\Big)\mathbb{P}_{\geq\lambda_{q+1}/10}\Big(\mathbb{W}_{(\overline{\xi})}\cdot\mathbb{W}_{(\overline{\xi}')}\Big)\\&\quad-a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\Big(\Big(W_{(\overline{\xi})}\cdot W_{(\overline{\xi}')}\Big)\nabla\Big(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\Big)\Big),\end{aligned}$$

where the second term is a pressure, the third can be estimated analogously to $E_{(\overline{\xi},\overline{\xi}',1)}$. Also note that the first and fourth term can estimated analogously. Using (16), (34) and (35), we obtain

$$\Big\|\mathcal{R}\Big(a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\Big(\nabla\Big(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\Big)\cdot\Big(W_{(\overline{\xi})}\otimes W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\otimes W_{(\overline{\xi})}\Big)\Big)\Big)\Big\|_{L^\infty_tL^p_x}$$

$$
\begin{aligned}
&\lesssim \lambda_{q+1}^{-1}\left\|a_{(\overline{\xi})}a_{(\overline{\xi}')}\right\|_{C^3}\left\|\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)\right\|_{L_t^\infty L_x^p}\\
&\lesssim \sigma r^{4-3/p}\mathcal{C}_3.
\end{aligned}
$$

Now consider $E_{(\overline{\xi},-\overline{\xi},2)}$. We can write

$$
\begin{aligned}
\nabla\cdot\left(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(-\overline{\xi})}+\mathbb{W}_{(-\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi})}\right) &= \left(W_{(-\overline{\xi})}\cdot\nabla\eta^2_{(\overline{\xi})}\right)W_{(\overline{\xi})}+\left(W_{(\overline{\xi})}\cdot\nabla\eta^2_{(\overline{\xi})}\right)W_{(-\overline{\xi})}\\
&= \left(A_{\overline{\xi}}\cdot\nabla\eta^2_{(\overline{\xi})}\right)A_{\overline{\xi}}+\left((\overline{\xi}\times A_{\overline{\xi}})\cdot\nabla\eta^2_{(\overline{\xi})}\right)\left(\overline{\xi}\times A_{\overline{\xi}}\right)\\
&= \nabla\xi^2_{(\overline{\xi})}-\left(\overline{\xi}\cdot\nabla\eta^2_{(\overline{\xi})}\right)\overline{\xi}\\
&= \nabla\eta^2_{(\overline{\xi})}-\frac{\overline{\xi}}{\mu}\partial_t\eta^2_{(\overline{\xi})},
\end{aligned}
$$

where we use (11) and the fact that $\{\overline{\xi}, A_{\overline{\xi}}, \overline{\xi}\times A_{\overline{\xi}}\}$ forms an orthonormal basis of $\mathbb{R}^3$. Therefore, we can write

$$
\begin{aligned}
E_{(\overline{\xi},-\overline{\xi},2)} &= \mathbb{P}_{\neq 0}\left(a^2_{(\overline{\xi})}\nabla\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}\eta^2_{(\overline{\xi})}-a^2_{(\overline{\xi})}\frac{\overline{\xi}}{\mu}\partial_t\eta^2_{(\overline{\xi})}\right)\\
&= \nabla\left(a^2_{(\overline{\xi})}\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}\eta^2_{(\overline{\xi})}\right)-\mathbb{P}_{\neq 0}\left(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}(\eta^2_{(\overline{\xi})})\nabla a^2_{(\overline{\xi})}\right)\\
&\quad -\mu^{-1}\partial_t\mathbb{P}_{\neq 0}\left(a^2_{(\overline{\xi})}\eta^2_{(\overline{\xi})}\overline{\xi}\right)+\mu^{-1}\mathbb{P}_{\neq 0}\left(\partial_t\left(a^2_{(\overline{\xi})}\right)\eta^2_{(\overline{\xi})}\overline{\xi}\right).
\end{aligned}
$$

Using the identity $\mathrm{Id}-\mathbb{P}_{LH}=\nabla\Delta^{-1}\mathrm{div}$, we obtain

$$
\begin{aligned}
\sum_{\overline{\xi}}E_{(\overline{\xi},-\overline{\xi},2)}+\partial_t w^{(t)}_{q+1} &= \nabla\sum_{\overline{\xi}}\left(a^2_{(\overline{\xi})}\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}\eta^2_{(\overline{\xi})}\right)-\nabla\sum_{\overline{\xi}}\mu^{-1}\Delta^{-1}\nabla\cdot\partial_t\left(a^2_{(\overline{\xi})}\eta^2_{(\overline{\xi})}\overline{\xi}\right)\\
&\quad -\sum_{\overline{\xi}}\mathbb{P}_{\neq 0}\left(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}(\eta^2_{(\overline{\xi})})\nabla a^2_{(\overline{\xi})}\right)+\mu^{-1}\sum_{\overline{\xi}}\mathbb{P}_{\neq 0}\left(\partial_t\left(a^2_{(\overline{\xi})}\right)\eta^2_{(\overline{\xi})}\overline{\xi}\right),
\end{aligned}
$$

where the first and second terms are pressure terms. Using (16), (34) and (35), we obtain

$$
\begin{aligned}
\|\mathcal{R}\mathbb{P}_{\neq 0}\left(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}(\eta^2_{(\overline{\xi})})\nabla a^2_{(\overline{\xi})}\right)\|_{L_t^\infty L_x^p} &\lesssim (\lambda_{q+1}\sigma)^{-1}\|\eta_{(\overline{\xi})}\|^2_{L_t^\infty L_x^{2p}}\mathcal{C}_3\\
&\lesssim (\lambda_{q+1}\sigma)^{-1}r^{3-3/p}\mathcal{C}_3.
\end{aligned}
$$

It follows from (16) and (34) that

$$
\begin{aligned}
\mu^{-1}\|\mathcal{R}\mathbb{P}_{\neq 0}\left(\partial_t\left(a^2_{(\overline{\xi})}\right)\eta^2_{(\overline{\xi})}\overline{\xi}\right)\|_{L_t^\infty L_x^p} &\lesssim \mu^{-1}\|\partial_t\left(a^2_{(\overline{\xi})}\right)\eta^2_{(\overline{\xi})}\overline{\xi}\|_{L_t^\infty L_x^p}\\
&\lesssim \mu^{-1}r^{3-3/p}\mathcal{C}_1.
\end{aligned}
$$

Let us now give the explicit definition of $\widetilde{R}_{oscillation}$:

$$
\begin{aligned}
\widetilde{R}_{oscillation} &= \sum_{\overline{\xi},\overline{\xi}'\in\Lambda}\mathbb{P}_{\neq 0}\left(\nabla(a_{(\overline{\xi})}a_{(\overline{\xi}')})\cdot(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}+\mathbb{W}_{\overline{\xi}'}\otimes\mathbb{W}_{(\overline{\xi})}))\right)\\
&\quad +\sum_{\overline{\xi},\overline{\xi}'\in\Lambda,\overline{\xi}\neq\overline{\xi}'}a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)\cdot\left(W_{(\overline{\xi})}\otimes W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\otimes W_{(\overline{\xi})}\right)\right)\\
&\quad -\sum_{\overline{\xi},\overline{\xi}'\in\Lambda,\overline{\xi}\neq\overline{\xi}'}\nabla\left(a_{(\overline{\xi})}a_{(\overline{\xi}')}\right)\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\mathbb{W}_{(\overline{\xi})}\cdot\mathbb{W}_{(\overline{\xi}')}\right)
\end{aligned}
$$

$$- \sum_{\overline{\xi},\overline{\xi}'\in\Lambda,\overline{\xi}\neq\overline{\xi}'} a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\Big(\Big(W_{(\overline{\xi})}\cdot W_{(\overline{\xi}')}\Big)\nabla\Big(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\Big)\Big)$$
$$- \sum_{\overline{\xi}\in\Lambda}\mathbb{P}_{\neq 0}\Big(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}(\eta^2_{(\overline{\xi})})\nabla a^2_{(\overline{\xi})}\Big) + \mu^{-1}\sum_{\overline{\xi}\in\Lambda}\mathbb{P}_{\neq 0}\Big(\partial_t\Big(a^2_{(\overline{\xi})}\Big)\eta^2_{(\overline{\xi})}\overline{\xi}\Big).$$

Finally, we estimate the time support of R_{q+1}. Using (25) we obtain

$$\operatorname{supp}_t R_{q+1} \subset \operatorname{supp}_t w_{q+1} \cup \operatorname{supp}_t R_q \subset N_{\delta_{q+1}}(\operatorname{supp}_t R_q).$$

Now we choose the parameters r, σ, μ. Fix α so that

$$\max\left\{0, \frac{2}{3}(2\theta - 1)\right\} < \alpha < 1,$$

which is possible since $\theta \in (-\infty, 5/4)$. Fix

$$r = \lambda_{q+1}^{\alpha}, \quad \sigma = \lambda_{q+1}^{-(\alpha+1)/2}, \quad \mu = \lambda_{q+1}^{(5\alpha+1)/4}. \tag{37}$$

Clearly (27) is satisfied. Choose $p > 1$ sufficiently close to 1 so that

$$-\frac{\alpha+1}{2} + \frac{5\alpha+1}{4} + \left(\frac{5}{2} - \frac{3}{p}\right)\alpha < 0, \quad \left(\frac{3}{2} - \frac{3}{p}\right)\alpha + \max(0, 2\theta - 1) < 0,$$
$$-\frac{5\alpha+1}{4} + \left(\frac{9}{2} - \frac{3}{p}\right)\alpha < 0, \quad -\frac{1-\alpha}{2} + \left(3 - \frac{3}{p}\right)\alpha < 0.$$

Note that C_N is independent of λ_{q+1}, due to (24). Combining the above estimates with Lemma 4, it is easy to check that, by taking λ_{q+1} sufficiently large, we arrive at (4), (6) and (7). This completes the proof of Lemma 1.

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